Kuwait University | Math 101 | January 3, 2001
Dept of Math & Comp Sci | Final Examination | Duration: 2 hours

Answer the following questions. Calculators, Mobile Phones and Pagers are not allowed

1. (a) Find $\lim_{x\to -3} \dfrac{1-\cos(x+3)}{x^2+x-6}$. (3 Points)

   (b) Let $f(x) = \begin{cases} 3x^2-2, & \text{if } x<0, \\ \dfrac{x^2-2x-3}{x^2-4x+3}, & \text{if } x \geqslant 0. \end{cases}$ Find all points of discontinuity of f and classify each discontinuity as removable, infinite or jump. (3 Points)

2. (a) Let $f(x) = x^3 + \frac{1}{\pi}\sec^2\left(\frac{\pi}{8}x\right)$. Use differentials to find the approximate change in f if x changes from 2 to 2.01. (3 Points)

   (b) Determine whether $f(x) = x^2 + 4\sqrt{5-x^2}$ satisfies the hypotheses of Rolle's theorem on $[-2,2]$, and if so, find the numbers c satisfying the conclusion of the theorem. (3 Points)

3. Find the points on the graph of $\dfrac{x^2}{4} - y^2 = 1$ that are closest to the point $P(5,0)$. (4 Points)

4. (a) Let $f(x) = x^3 + \sin x$. Find the average value of f on $[-3,3]$. (3 Points)

   (b) Find $f'(x)$ if $f(x) = \displaystyle\int_{3x}^{x^2} \sqrt{\sin^2 t + 7t^6}\, dt$. (3 Points)

5. Evaluate the following integrals (3 points each)

   (a) $\displaystyle\int \sqrt{x}(\sqrt{x}+\sqrt{3})\, dx$ (b) $\displaystyle\int_0^{\pi/4} \sin 2x \cos^7 2x\, dx$.

6. (a) Find the area of the region bounded by the graphs of the equations $y=x$ and $y=x^3$. (3 Points)

   (b) Find the arc length of the graph of $y = \dfrac{x^4}{16} + \dfrac{1}{2x^2}$ from $A(-2, \frac{9}{8})$ to $B(-1, \frac{9}{16})$. (3 Points)

7. Let R be the region bounded by the graphs of the equations $y = x^2+3$, $y=0$, $x=0$, and $x=2$. Setup an integral that can be used to find the volume of the solid generated by revolving R about

   (a) the x-axis. (3 Points)

   (b) the line $x=-5$. (3 Points)

Final Exam. Solution Key Math. 101 January 3, 2001

1. (a) $\lim_{x\to -3} \dfrac{1-\cos(x+3)}{x^2+x-6} = \lim_{(x+3)\to 0} \dfrac{1-\cos(x+3)}{(x+3)} \times \lim_{x\to -3}\left(\dfrac{1}{x-2}\right) = \boxed{0}$.

(b) $\lim_{x\to 0^+} f(x) = -1$, $\lim_{x\to 0^-} f(x) = -2 \Longrightarrow f$ has $\boxed{\text{jump}}$ discontinuity at $\boxed{x=0}$

$\lim_{x\to 3} f(x) = 2$, $f(3)$ is $UD \Longrightarrow f$ has $\boxed{\text{removable}}$ discontinuity at $\boxed{x=3}$.

$\lim_{x\to 1^+} f(x) = +\infty$, $\lim_{x\to 1^-} f(x) = -\infty \Longrightarrow f$ has $\boxed{\text{infinite}}$ discontinuity at $\boxed{x=1}$.

2. (a) $f'(x) = 3x^2 + \frac{1}{4}\sec^2\left(\frac{\pi x}{8}\right)\tan\left(\frac{\pi x}{8}\right)$, $f'(2) = \frac{25}{2}$, $\Delta x = 0.01$

$\Delta y \simeq dy = f'(2)(0.01) = \boxed{\frac{1}{8}}$.

(b) f is continuous on $[-2,2]$, f is differentiable on $(-2,2)$ and $f(-2) = 8 = f(2)$ then f satisfies the requirements of Rolle's theorem on $[-2,2]$,

$f'(x) = 2x - \dfrac{4x}{\sqrt{5-x^2}}$. From Rolle's theorem $\exists c \in (-2,2)$ such that $f'(c) = 0$, i.e.,

$2c - \dfrac{4c}{\sqrt{5-c^2}} = 0 \Longrightarrow \boxed{c = 0, \pm 1 \in (-2,2)}$.

3. Let D be the distance between $P(5,0)$ and the point (x,y) on the graph of the equation $\frac{x^2}{4} - y^2 = 1$. So,

$D^2 = f(x) = (x-5)^2 + (y-0)^2 = (x-5)^2 + \left(\frac{x^2}{4} - 1\right)$, $f'(x) = \frac{5}{2}x - 10$,

$f''(x) = \frac{5}{2} > 0$. Thus, f is minimaum at $x = 4, y = \pm\sqrt{3}$. And hence, D is minimaum at $x = 4, y = \pm\sqrt{3}$ i.e., at $\boxed{\left(4,\sqrt{3}\right) \text{ and } \left(4,-\sqrt{3}\right)}$.

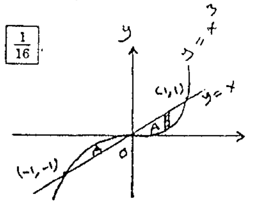
4. (a) $f_{av} = \frac{1}{6}\int_{-3}^{3} (x^3 + \sin x)\, dx = \boxed{0}$, since $(x^3 + \sin x)$ is odd function.

(b) $f'(x) = 2x\sqrt{\sin^2 x^2 + 7x^{12}} - 3\sqrt{\sin^2 3x + 7(3x)^6}$.

5. (a) $\int \left(x + \sqrt{3}\sqrt{x}\right) dx = \boxed{\dfrac{x^2}{2} + \dfrac{2\sqrt{3}}{3}x^{\frac{3}{2}} + C}$.

(b) Put $u = \cos 2x \Rightarrow du = -2\sin 2x\, dx$ and $u(0) = 1, u\left(\frac{\pi}{4}\right) = 0$,

$\int_0^{\pi/4} \sin 2x \cos^7 2x\, dx. = -\frac{1}{2}\int_1^0 u^7\, du = -\frac{1}{2}\left[\frac{u^8}{8}\right]_1^0 = \boxed{\frac{1}{16}}$



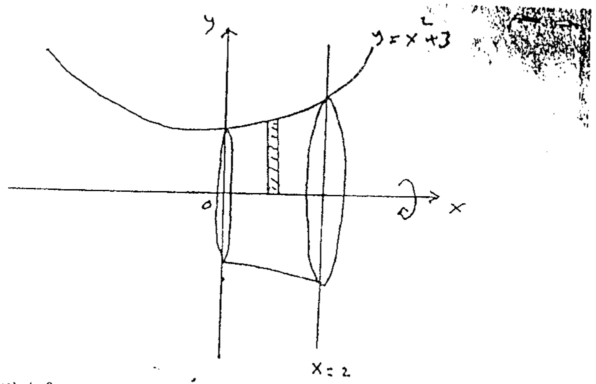
6. (a) $R = 2A = 2\int_0^1 (x - x^3)\, dx = \boxed{\frac{1}{2}}$.

(b) $y' = \dfrac{1}{4}x^3 - \dfrac{1}{x^3}$, $1 + (y')^2 = \left(\dfrac{1}{4}x^3 + \dfrac{1}{x^3}\right)^2$, $\sqrt{1+(y')^2} = \left|\dfrac{1}{4}x^3 + \dfrac{1}{x^3}\right| = -\left(\dfrac{1}{4}x^3 + \dfrac{1}{x^3}\right)$

$L_{-2}^{-1} = \int_{-2}^{-1} \sqrt{1+(y')^2}\,dx = -\int_{-2}^{-1}\left(\dfrac{1}{4}x^3 + \dfrac{1}{x^3}\right) dx = \left[\frac{x^4}{16} - \frac{1}{2x^2}\right]_{-1}^{-2} = \boxed{\frac{21}{16}}$.

7. (a) *Volume of disk* $= \pi\left(x^2+3\right)^2 dx$

$V = \pi \int_0^2 \left(x^2+3\right)^2 dx.$



(b) *Radius of cylindrical shell* $= x + 5$.

Volme of cylindrical shell $= 2\pi(x+5)\left(x^2+3\right) dx$

$V = 2\pi\int_0^2 (x+5)\left(x^2+3\right) dx.$